

A Note on Shared Randomness and Shared Entanglement in Communication

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Abstract

We consider several models of 1-round classical and quantum communication, some of these models have not been defined before. We “almost separate” the models of *simultaneous quantum message passing with shared entanglement* and the model of *simultaneous quantum message passing with shared randomness*. We define a relation which can be efficiently *exactly* solved in the first model but cannot be solved efficiently, *either exactly or in 0-error setup* in the second model. In fact, our relation is exactly solvable even in a more restricted model of *simultaneous classical message passing with shared entanglement*.

As our second contribution we strengthen a result by Yao which says that a “very short” protocol from the model of *simultaneous* classical message passing with shared randomness can be simulated in the model of simultaneous quantum message passing: for a boolean function f , $Q^{\parallel}(f) \in \exp(O(R_p^{\parallel}(f))) \cdot \log n$. We show a similar result for protocols from a (stronger) model of *1-way* classical message passing with shared randomness: $Q^{\parallel}(f) \in \exp(O(R_p^1(f))) \cdot \log n$. We demonstrate a problem whose efficient solution in Q^{\parallel} follows from our result but not from Yao’s.

1 Introduction

In this work we consider several models of 1-round communication, both classical and quantum (see Section 2 for definitions). To the best of our

knowledge, some of these models have never been studied before, nor were they defined. For example, we consider the situations when quantum communication channels are used but the players share a classical random string (and no entanglement), as well as the case when the players share entanglement but communication channels are classical.

We argue that these “unusual” models are interesting to look at. There are several open questions related to these models which can be viewed as natural analogues of yet unsolved questions related to the usual models. Besides, models like *entangled parties with classical communication channels* might correspond to some “realistic” circumstances, when physical constraints make it hard to establish quantum communication. We also note that the setup of entangled players sending simultaneous classical messages is closely related to the matters considered by Cleve et al. in [CHTW04].

Our main interest in this paper is the power of shared entanglement in communication; in particular, we are looking for similarities and differences between entanglement and the (intuitively similar) resource of shared randomness in the context of 1-round communication models. To the best of our knowledge, prior to this work there was no explicit construction showing that entanglement can be more powerful than shared randomness in the same model of communication (for that one has to deal with at least one model which is *unusual* in the above sense).

1.1 Our Results

In this work we “almost separate” the models of *simultaneous quantum message passing with shared entanglement* and the model of *simultaneous quantum message passing with shared randomness*. We define a relation MHM_n which can be solved *exactly* in the first model¹ but cannot be solved *either exactly or in 0-error setup* in the second model (where “solved” means efficient solution of cost $\text{poly}(\log n)$). We *conjecture* that our relation is hard for the latter model even in the bounded error case.

Besides, we extend a result by Yao who has shown that a “very short” protocol from the model of *simultaneous* classical message passing with shared randomness can be simulated in the model of simultaneous quantum message passing (with no shared resource): for a boolean function f , $Q^\parallel(f) \in 2^{O(R_p^\parallel(f))} \cdot \log n$. We show a similar result for protocols from a (stronger) model

¹Actually, we show that MHM_n is solvable in a more restricted model R_e^\parallel .

of *1-way* classical message passing with shared randomness: for a boolean function f , $Q^\parallel(f) \in 2^{O(R_p^1(f))} \cdot \log n$. We demonstrate a communication problem whose efficient solution in Q^\parallel follows from our simulation technique but not from Yao's.

As a straightforward corollary, we show that a protocol of constant cost in the model of *1-way quantum* message passing with shared randomness can be efficiently simulated in the model of simultaneous quantum message passing.

2 Notation

In this paper we will consider several models of single round communication between two parties (*Alice* and *Bob*). In general, a communication task can be vied as follows: Alice receives an input string x , Bob receives an input string y , then some communication occurs which allows to compute output, based on x and y . The goal is to produce “good” output using minimum amount of communication (measured in either bits or qubits). Communication task defines which outputs are good for every possible input, this task can be given in terms of either a function or a relation (the latter allows several good outputs for each input).

A *communication model* defines what sort of communication can be performed in order to solve the problem. We will consider two types of models. In *simultaneous message passing* models Alice and Bob send one message each to the third party (a *referee*), who has to produce an output based on the received messages (the model is called simultaneous since Alice and Bob do not receive any information from each other and therefore they can produce their messages simultaneously, or more precisely, asynchronously). In *1-way communication* models Alice sends a message to Bob and he has to produce an output based on that message and his part of the input.

Sometimes an additional resource is given to Alice and Bob in order to reduce the communication cost, that can be either a string of shared random bits or shared pairs of entangled qubits (w.l.g., EPR pairs). In either case, the amount of the shared resource is not limited by the model.

We will be interested in the following models of communication:

Definition 1. *Let*

$$R^1, R_p^1, R_e^1, R^\parallel, R_p^\parallel, R_e^\parallel, Q^1, Q_p^1, Q_e^1, Q^\parallel, Q_p^\parallel, Q_e^\parallel$$

be 1-round communication models defined as follows:

- \mathbf{R} corresponds to communication using classical channels and \mathbf{Q} corresponds to communication through quantum channels;
- the superscript \parallel corresponds to simultaneous message passing and $\mathbf{1}$ corresponds to 1-way communication;
- the subscript \mathbf{p} means that a “public” random string is shared between two communicating parties (two broadcasting parties in case of simultaneous message passing), and the subscript \mathbf{e} means that EPR pairs are shared between the parties.

We will consider communication complexity of both functions and relations. We require that a protocol produces right answer with *constant advantage over trivial* (where “trivial” usually means achievable by a random guess, e.g., nontrivial accuracy is any constant greater than $1/2$ in the case of binary functions, while any constant greater than 0 is nontrivial in the case of 0-error protocols).

Given a communication task P (either a function or a relation), we will write $R^1(P)$, etc. to address the number of (qu)bits transferred by the “cheapest” protocol solving P in the corresponding model. We will write $P \in R^1$ when $R^1(P) \in \text{poly}(\log n)$, and similarly for the rest of defined models.

We will use sign \subset to denote proper sets inclusion and by \subseteq we will denote the regular inclusion.

3 Classical Randomness in Communication

Quantum communication protocol with (classical) random coins (in both Q_p^1 and Q_p^\parallel) can be naturally viewed as a “classical mixture” of quantum protocols, where the random string is replaced with a predefined binary sequence.²

That is why several classical (in both senses) results readily extend to the models Q_p^1 and Q_p^\parallel . Let us mention two of them, which we will need later.

²In the case of Q_p^\parallel there exists one obstacle: the referee does not know the random string shared between two broadcasting parties. However we will see that $O(\log n)$ bits of shared randomness are sufficient, so the referee can receive their values from one of the parties, at the price of additional $O(\log n)$ communicated qubits.

Fact 2. ([N91]) *Communication protocols in the models with shared randomness can w.l.g. be assumed to be using $O(\log n)$ bits of shared randomness.*

In particular, it follows that $R_p^I = R^I$, because Alice can send to Bob $O(\log n)$ random bits she used, thus making them “public”.

By *distributional deterministic* communication complexity of a problem with respect to input distribution D we mean the minimum cost of a deterministic protocol solving the problem with constant advantage over trivial success probability, when the input is distributed according to D .

Fact 3. ([Y83]) *Communication cost of a problem in the models with shared randomness is lower bounded by the distributional deterministic complexity of the same problem in the same model, with respect to any input distribution.*

Actually, Fact 3 is one direction (the “easy” one) of the Minimax Theorem.

4 1-Way Communication

In this section we give two simple equivalence results for the 1-way communication models we defined.

The first equivalence follows from Fact 2:

Proposition 4. $Q_p^I = Q^I$.

Similarly to the case of classical communication channels, Alice can send to Bob $O(\log n)$ random bits, thus making them “public”.

The second equivalence is a consequence of the quantum teleportation phenomena:

Proposition 5. $R_e^I = Q_e^I$.

Using shared EPR pairs, Alice can teleport to Bob through the classical channel her quantum message.

5 Simultaneous Message Passing

Usually the setting of simultaneous message passing is more interesting (and sometimes harder to analyze) than 1-way communication. For example, in the classical theory of communication complexity the model of simultaneous messages is the only one which can be noticeably strengthened by adding to it shared randomness.

5.1 A Generalization of Yao's Simulation

In [Y03] Yao shows that a protocol of constant complexity in R_p^\parallel can be simulated by a protocol of complexity $O(\log n)$ in Q^\parallel :

Fact 6. ([Y03]) *Let f be a boolean function. Then*

$$Q^\parallel(f) \in 2^{O(R_p^\parallel(f))} \cdot \log n.$$

Let us generalize his result. We do that in two steps.

Proposition 7. *For any boolean function f ,*

$$Q^\parallel(f) \in 2^{O(R_p^1(f))} \cdot \log n.$$

Remark 1: Note that in the context of Proposition 7 the communication models R_p^1 and R^1 are *not equivalent*, since additive factor $\log n$ becomes significant (cf. with Fact 2).

Proof of Proposition 7. Let s be the communication cost of a protocol for $f(x, y)$ in R_p^1 which uses $r \in O(\log n)$ public random bits and is correct with at least constant probability higher than $1/2$. Let $a(x, q)$ be the message sent by Alice in this protocol when her input is x and the random string is q . Let $b(y, a, q)$ be a boolean predicate ($\{0, 1\}$ -valued) getting value 1 if Bob accepts given input y , random string q and the message from Alice being a .

To “simulate” the original protocol in Q^\parallel , do the following (for $k \in \mathbb{N}$ and $0 < \tau < 1$ to be chosen later):

- Alice sends k copies of

$$|\alpha\rangle \stackrel{\text{def}}{=} 2^{-\frac{r}{2}} \cdot \sum_q |q\rangle |a(x, q)\rangle |1\rangle.$$

- Bob sends k copies of

$$|\beta\rangle \stackrel{\text{def}}{=} 2^{-\frac{r+s}{2}} \cdot \sum_{q,a} |q\rangle |a\rangle |b(y, a, q)\rangle.$$

- Using the swap test, the referee estimates the value of $\langle\alpha|\beta\rangle$ and accepts if the approximation is at least τ .

Observe that

$$\begin{aligned} \langle\alpha|\beta\rangle &= 2^{-(r+\frac{s}{2})} \sum_{q,a} \langle q|q\rangle \langle a(x, q)|a\rangle \langle 1|b(y, a, q)\rangle \\ &= 2^{-\frac{s}{2}} \mathbf{E}_q [I(q)], \end{aligned}$$

where $I(q)$ is an indicator of the event *given the random string q , Alice sends a message which causes Bob to accept*. In other words, $I(q)$ gets value 1 if and only if the content q of the shared random string would cause the original protocol to accept.

Therefore, if we set $\tau = 2^{-(s/2+1)}$ and $k \in 2^{O(s)}$ high enough, with high confidence the new protocol would accept (x, y) if and only if the original one would do so with probability greater than $1/2$. The cost of the new protocol is $2^{O(s)} \cdot \log n$, as required. ■ Proposition 7

Proposition 8. *For any function f ,*

$$R_p^1(f) \in 2^{O(Q_p^1(f))}.$$

Proof sketch of Proposition 8. Assume w.l.g. that the messages from Alice are pure states (this can be achieved by at most doubling the number of communicated qubits). Since we only require constant precision, we can replace the quantum message from Alice by a classical one of exponential length. ■ Proposition 8

Based on Propositions 7 and 8, we get the following corollary:

Corollary 9. *A function f of constant communication complexity in Q_p^1 can be solved in Q^{\parallel} using $O(\log n)$ qubits of communication.*

5.1.1 Strength of Our Improvement

We think that Proposition 8 and Corollary 9 are not very interesting from the technical point of view. In particular, Corollary 9 might be established without Proposition 7, based on the (trivial) statement that *for any boolean f , $R_p^\parallel(f) \in 2^{O(R_p^1(f))}$ and on the original Fact 6.*

On the other hand, Proposition 7 is less trivial. Note that if we would try to establish a similar result through application of Fact 6 and simulating an R_p^1 -protocol in R_p^\parallel , the price we would pay for such simplification would be exponential loss in tightness (we would end up with something like $Q^\parallel(f) \in \exp(\exp O(R_p^1(f))) \cdot \log n$).

To establish the “usefulness” of Proposition 7 let us define a function f such that

$$R_p^1(f) \in O(\log(\log n))$$

but

$$R_p^\parallel(f) \in \Omega(\log n).$$

In other words, membership of f in Q^\parallel would follow from Proposition 7, but not from Yao’s original Fact 6.

We first define an “auxiliary” predicate.

Definition 10. *Let $x \in \{0, 1\}^{\log \log m}$ and $y = (y_1, \dots, y_{(\log m)/2})$, where all y_i -s are distinct binary strings of length $\log \log m$. Then $\text{sub}_m(x, y) = 1$ if and only if x is identical to one of y_i -s.*

Definition 11. *Let $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$, where each pair (a_i, b_i) forms a correct input to sub_m . Then $f(a, b) = 1$ if $|\{i \mid \text{sub}_m(a_i, b_i)\}| \geq m/2$ and $f(a, b) = 0$ if $|\{i \mid \text{sub}_m(a_i, b_i)\}| = 0$, with a promise that one of the two cases holds.*

As usual, we use n to denote the length of input to f .

To see that $R_p^1(f) \in O(\log(\log n))$, consider a protocol where several random indices in the range $\{1, \dots, m\}$ are tossed as a public coin, then Alice sends a_i -s corresponding to those random indices and Bob accepts if and only if for at least one of the indices $\text{sub}_m(a_i, b_i)$ is satisfied.

To show that $R_p^\parallel(f) \in \Omega(\log n)$ we use an approach suggested by Fact 3, we will fix the input distribution and restrict our attention to deterministic protocols. Denote by X the set of (a, b) -s which form correct input to f and in every $a = (a_1, \dots, a_m)$ all a_i -s are identical and in every $b = (b_1, \dots, b_m)$ all

b_i -s are identical. Then according to our distribution, the input is with probability $1/2$ a uniformly chosen positive instance from X and with probability $1/2$ a uniformly chosen negative instance from X .

Under this distribution the actual task of a protocol solving f would be to solve an instance of $\text{sub}_m(a_0, b_0)$ when Alice receives a_0 and Bob receives b_0 . Our input distribution for $\text{sub}_m(a_0, b_0)$ corresponds to uniformly choosing a positive instance with probability $1/2$ and a negative instance with probability $1/2$.

Let us see that this task requires $\Omega(\log n)$ communication in the model of R^\parallel , when the protocol is deterministic (and the error is bounded by a constant smaller than $1/2$). W.l.g., let Alice always send a_0 to the referee. Then it is “easy to see” that Bob has to send $\Omega(\log n)$ bits, since his part of input is a random subset of $\{1, \dots, m\}$ of size $m/2$ and his message to the referee should enable the recipient to decide with constant accuracy whether a random a_0 belongs to that subset.

Remark 2: Note that we could use a “padded version” of sub_m as a communication task whose membership in Q^\parallel follows from Proposition 7, but not from Fact 6. However, the example of f is probably more interesting, since for its efficient solution shared randomness is necessary (while for solving sub_m shared randomness is not required).

Remark 3: There exists another qualitative difference between our Proposition 7 and Yao’s Fact 6: while the latter easily extends to the case of *relational problems* ([GKW04]), the bound obtained in Proposition 7 seems to be “inherently functional” (even boolean, in some sense).

5.2 Separating Q_e^\parallel from Q_p^\parallel and R_e^\parallel from R_p^\parallel

In this section we “almost show” that $R_e^\parallel \not\subseteq Q_p^\parallel$ (and therefore $Q_p^\parallel \subset Q_e^\parallel$). We demonstrate a relation which can be solved *exactly* in R_e^\parallel (and therefore in Q_e^\parallel), but cannot be efficiently solved in Q_p^\parallel in the following setting: the referee may not make a mistake, however he is allowed to announce *don’t know* with some constant probability (less than 1). Of course, this requirement (we call it “*don’t know*” setting) is less severe than exact solvability, and therefore our result may be viewed as exponential separation between the model of quantum communication with shared entanglement and the model of quantum communication with shared randomness.

On the other hand, we *conjecture* that our relation is hard for the “standard” Q_p^\parallel (which we address as *bounded-error setting*), but we could not prove that. However, there is a known lower bound for the bounded-error setting for the model R^1 (and therefore for R_p^\parallel) given in [BJK04] for a communication problem which is actually a simplified (and easier for communication) version of the relation we will define here. So, we conclude that $R_p^\parallel \subset R_e^\parallel$ and even that $R_e^\parallel \not\subseteq R^1$ in the standard bounded-error setting.³

Generalizing a construction used in [BJK04], we define a family of relations parametrized by $\{n \in \mathbb{N} \mid n \text{ is even}\}$. Denote by M_n the set of all perfect matchings over n elements (labeled as $\{1, \dots, n\}$).

Definition 12. Let $a \in \{0, 1\}^n$ and $m \in M_n$. For any $x = (a^{(A)}, m)$ and $y = a^{(B)}$ such that $a^{(A)} \oplus a^{(B)} = a$ (where \oplus means bit-wise xor),

$$MHM_n(x, y) = \{(i, j, b) \mid a_i \oplus a_j = b, (i, j) \in m\}.$$

In order to show that MHM_n is efficiently exactly solvable in R_e^\parallel we adapt a clever protocol suggested by Buhrman [B] (see also [BBT04]).

5.2.1 MHM_n Is Exactly Solvable in R_e^\parallel

Consider the following R_e^\parallel -protocol for MHM_n .

- Before the communication starts, Alice and Bob share $\lceil \log n \rceil$ pairs of entangled qubits: $\sum_{i \in [n]} |i\rangle |i\rangle$.
- When Alice receives $x = (a^{(A)}, m)$ she applies the following transformation to her part of the entangled pairs:

$$|i\rangle \rightarrow (-1)^{a_i^{(A)}} |i\rangle.$$

Similarly, Bob flips the sign of those parts of the entangled $\sum |i\rangle |i\rangle$ which correspond to $a_i^{(B)} = 1$.

- Alice performs partial projection of her part of entangled qubits to the subspaces of dimension 2 spanned by all pairs of indices in the matching m (fulfilled if necessary by “insignificant” subspace of the

³The last observation is actually not a contribution of this paper but rather a *compilation* of lower and upper bounds from [BJK04] and [B], correspondingly.

indices greater than n , less or equal to $2^{\lceil \log n \rceil}$). After the measurement, the common state of shared entangled pairs becomes

$$\frac{1}{\sqrt{2}}(|k\rangle|k\rangle \pm |l\rangle|l\rangle),$$

where $(k, l) \in m$. Moreover, the amplitudes of $|k\rangle|k\rangle$ and $|l\rangle|l\rangle$ coincide if and only if $a_k = a_l$.

- Both Alice and Bob perform $\lceil \log n \rceil$ -qubit Hadamard transform on their parts of entangled qubits and then measure in the standard basis. As a result, Alice obtains b_1 and Bob obtains b_2 , such that

$$(b_1 \oplus b_2) \cdot (k \oplus l) = a_k \oplus a_l, \quad (1)$$

where \oplus denotes bit-wise xor operation and \cdot stands for the inner product *mod 2* of two vectors. Alice sends (k, l, b_1) and Bob sends b_2 to the referee.

- According to (1), the referee computes $a_k \oplus a_l$ and outputs the triple $(k, l, a_k \oplus a_l)$, as required.

The protocol and the analysis are very similar to those in [BJK04]. The probability that the referee produces (k, l, b) is

$$\left| \frac{1}{\sqrt{2}}(\langle k| + (-1)^b \langle l|) |\varphi\rangle \right|^2 = \frac{1}{2n}((-1)^{a_k} + (-1)^{a_l+b})^2,$$

which equals $2/n$ if $(k, l, b) \in MHM_n(x, y)$ and 0 otherwise. The protocol is always correct.

The fact that $MHM_n \in R_e^\parallel$ (which is strengthening of the more obvious $MHM_n \in Q_e^\parallel$) is due to [B].

5.2.2 $MHM_n \notin Q_p^\parallel$ for the “Don’t Know” Setting

Let us slightly simplify the task and assume that the matching m in the input to MHM_n always comes from a fixed family M'_n of $n/2$ edge-disjoint matchings on n elements (as before, n is even). Now we use Fact 3 to “get rid” of the shared randomness. We choose the input distribution D_1 to be uniform: $a^{(A)}, a^{(B)} \in \{0, 1\}^n$, $m \in M'_n$.

We will prove the following.

Claim 13. *The communication cost of computing MHM_n in the model Q^\parallel in the “don’t know” setting is $\Omega(n^{1/6})$, when the input distribution is D_1 .*

The claim leads to the following theorem.

Theorem 14. *The communication cost of computing MHM_n in the model Q_p^\parallel in the “don’t know” setting is $\Omega(n^{1/6})$.*

To prove the claim, we generalize a technique from [GKW04].

Proof of Claim 13. Consider a protocol T solving the problem in the “don’t know” setting with success probability at least $7/8$, let s be its communication cost (note that in the “don’t know” setting the success probability can be amplified using parallel repetition).

We know that $|M'_n| = n/2$, denote by M_1 the subset of those $m \in M'_n$ satisfying $\Pr_{D_1}[T \text{ fails} | m] \leq 1/4$, where the conditioning is on the fact that m is the matching given to Alice. It must hold that

$$|M_1| \geq n/4.$$

We want to claim that for some matching $m \in M_1$ the referee outputs every pair from m with rather low probability, unless s is big.

Let us forget for a moment about the cost of communication between Alice and the referee, and assume that Alice forwards her complete input to the referee (this setup corresponds to the 1-way communication model). Then the referee can output a pair $(i, j) \in m$ if and only if he knows the value of $a_i^{(B)} \oplus a_j^{(B)}$.

Consider the mixed state corresponding to the uniform distribution of $a^{(B)}$ (as imposed by D_1), and let β_x be the density matrix of the message sent by Bob to the referee when $a^{(B)} = x$. Then the whole message sent to the referee is

$$\beta = \frac{1}{2^n} \sum_x \beta_x.$$

Let us denote by $p(i, m)$ the probability that the referee outputs the pair $(i, j) \in m$ when he receives β from Bob and the required matching is $m \in M_1$. Let

$$\lambda_m \stackrel{\text{def}}{=} \max_i p(i, m)$$

be the highest probability for outputting any specific pair when the required matching is m . Denote $\lambda_0 = \min_m \lambda_m$. For each $m \in M_1$ let i_m be any fixed index satisfying $p(i_m, m) \geq \lambda_0$ and j_m be such that $(i_m, j_m) \in m$.

Suppose now that Bob sends $1/\lambda_0$ independent copies of his message, then the referee is able to learn the value of $a_{i_m}^{(B)} \oplus a_{j_m}^{(B)}$ with probability at least $1/2$. In other words, by sending s/λ_0 qubits Bob lets the referee learn any $\{a_{i_m}^{(B)} \oplus a_{j_m}^{(B)} \mid m \in M_1\}$ with probability at least $1/2$ (since Bob does not know m , his message must be good for any $m \in M_1$). Let $E = \{(i_m, j_m) \mid m \in M_1\}$ and V be the set of endpoints of E . The size of E is at least $n/4$ (since we have required that M_1 consists of disjoint matchings) and $|V| \geq \sqrt{n}/2$. Let $V' \subset V$ be a subset of size $\leq |V|/2$, such that every point in $V \setminus V'$ has at least one neighbor in V' (the neighborhood is defined by E).

Let us further strengthen the referee and let him know the values of all $\{a_i^{(B)} \mid i \in V'\}$. Then by choosing one of $m \in M_1$ and trying to learn the corresponding value of $a_{i_m}^{(B)} \oplus a_{j_m}^{(B)}$ using the original protocol, the referee is able to get (with certainty) the value of any $\{a_i^{(B)} \mid i \in V \setminus V'\}$ with probability at least $1/2$, using the message of length s/λ_0 . If instead of announcing “don’t know” the referee would make a random guess of the value of $a_i^{(B)}$, his total correctness probability would increase to $3/4$. Since $|V \setminus V'| \geq |V|/2 \geq \sqrt{n}/4$, by applying Nayak’s bound on the length of a random access codes ([N99]) we obtain the following relation: $s/\lambda_0 \geq (1 - H(3/4)) \cdot \sqrt{n}/4$, or

$$\lambda_0 \leq \frac{22s}{\sqrt{n}}.$$

Fix m_0 to be some matching from M_1 satisfying $\max_i p(i, m_0) \leq \frac{22s}{\sqrt{n}}$.

Now let us return to the simultaneous message model. Denote by D_2 the following input distribution: Alice receives $(a^{(A)}, m_0)$ and Bob receives $a^{(B)}$, where $a^{(A)}$ and $a^{(B)}$ are uniformly and independently distributed over $\{0, 1\}^n$. As follows from the definition of M_1 , the protocol T solves MHM_n over D_2 with success probability at least $3/4$, and every individual pair from m_0 is returned by T with probability not higher than

$$p_0 \stackrel{\text{def}}{=} \frac{22s}{\sqrt{n}}.$$

Fix an indexing for the edges of the matching we’ve chosen: $m_0 = \{e_k \mid 1 \leq k \leq n/2\}$. For the rest of the proof we will use the following notation:

$$x_k^{(A)} = a_{i_k}^{(A)} \oplus a_{j_k}^{(A)}, \quad x_k^{(B)} = a_{i_k}^{(B)} \oplus a_{j_k}^{(B)},$$

where $1 \leq k \leq n/2$ and $e_k = (i_k, j_k)$. In these terms, given an input from the support of the distribution D_2 , if the protocol is successful it should produce $x_k^{(A)} \oplus x_k^{(B)}$ together with $e_k = (i_k, j_k)$. If the input is distributed according to D_2 then the protocol succeeds with probability at least $3/4$ and each e_k is returned with probability at most p_0 .

The latter statement can be reformulated in a stronger form: the information received by the referee *does not allow it* to learn with 0-error any of e_k with probability higher than p_0 (this is true since from the beginning we might restrict attention to the protocols which always try to “boost” one of the edges). Let $q_k^{(A)}$ denote the success probability of the referee in guessing $x_k^{(A)}$ in the “don’t know” setting, when $a^{(A)}$ is uniformly chosen. Then according to the Holevo bound,

$$\sum_{1 \leq k \leq n/2} q_k^{(A)} \leq s. \quad (2)$$

Similarly define $q_k^{(B)}$, then

$$\sum_{1 \leq k \leq n/2} q_k^{(B)} \leq s. \quad (3)$$

Now we need the following lemma, which is a simple modification of Lemma 1 from [GKW04] (a proof can be found in the appendix section).

Lemma 15. *Consider two quantum registers, the first containing either α_0 or α_1 and the second containing either β_0 or β_1 (each register contains either one of the corresponding mixed states with probability $1/2$ and the registers are not correlated in any way). Denote by a the maximum success probability of distinguishing α_0 and α_1 in the “don’t know” setting, define b similarly for β_0 and β_1 . Let p be the maximum success probability for a measurements with 3 outcomes: $[\alpha_0 \otimes \beta_0 \text{ or } \alpha_1 \otimes \beta_1]$, $[\alpha_0 \otimes \beta_1 \text{ or } \alpha_1 \otimes \beta_0]$ and [don’t know] (where “successful” are the first two outcomes). Then $p \leq 4ab$.*

According to the lemma, the probability of the protocol to be successful is upper bounded by

$$\sum_{1 \leq k \leq n/2} 4q_k^{(A)} q_k^{(B)},$$

subject to (2), (3), and $q_k^{(A)} q_k^{(B)} \leq p_0$ for all $1 \leq k \leq n/2$ (which follows from the obvious lower bound ab on the success probability in simultaneous 0-error guessing of the content of the both registers in the condition of

Lemma 15). The maximum is achieved if we pick a set K of size $s/\sqrt{p_0}$ and fix $q_k^{(A)} = q_k^{(B)} = \sqrt{p_0}$ for every $k \in K$. This gives the following upper bound on the success probability:

$$\frac{s}{\sqrt{p_0}} \cdot 4p_0 = 4s\sqrt{p_0} \in O\left(\frac{s^{3/2}}{n^{1/4}}\right),$$

and its value must be at least $3/4$. This leads to $s \in \Omega(n^{1/6})$. ■ *Claim 13*

6 Open Problems

- We know that $Q^I = Q_p^I \subseteq Q_e^I = R_e^I$. Are they equal?
- We were only able to demonstrate that $MHM_n \notin Q_p^\parallel$ for “don’t know” setting, while we conjecture that the problem is hard for Q_p^\parallel in the standard (bounded-error) setting as well.
- We have shown our separation using a relation. Can similar results be obtained for a (partial) boolean function? What about a total function?

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A Appendix

A.1 Proof of Lemma 15

Lemma 15. *Consider two quantum registers, the first containing either α_0 or α_1 and the second containing either β_0 or β_1 (each register contains either one of the corresponding mixed states with probability $1/2$ and the registers are not correlated in any way). Denote by a the maximum success probability of distinguishing α_0 and α_1 in the “don’t know” setting, define b similarly for β_0 and β_1 . Let p be the maximum success probability for a measurements with 3 outcomes: $[\alpha_0 \otimes \beta_0 \text{ or } \alpha_1 \otimes \beta_1]$, $[\alpha_0 \otimes \beta_1 \text{ or } \alpha_1 \otimes \beta_0]$ and [don’t know] (where “successful” are the first two outcomes). Then $p \leq 4ab$.*

Proof of Lemma 15. In order to prove the lemma it suffice to slightly modify the proof of similar Lemma 1 in [GKW04].

Let us define several subspaces of our quantum registers. Call S_0 the support of α_0 (i.e., the span of its non-zero eigenvectors), and let S_1 be the support of α_1 . Similarly define T_0 and T_1 for β_0 and β_1 .

According to 0-error requirement of our lemma, the positive semidefinite operator corresponding to the measurement outcome $[\alpha_0 \otimes \beta_0 \text{ or } \alpha_1 \otimes \beta_1]$ should have all its eigenvectors lying inside $R \stackrel{\text{def}}{=} (S_1 \times T_0)^\perp \cap (S_0 \times T_1)^\perp$. Expanding, we get $R = (S_1^\perp \times T_1^\perp) \oplus (S_0^\perp \times T_0^\perp)$ (where \oplus denotes direct sum of vector spaces), so that the probability to observe this outcome is at most the probability to project the measured state to either $S_1^\perp \times T_1^\perp$ or $S_0^\perp \times T_0^\perp$.

Similarly, the probability to obtain the outcome $[\alpha_0 \otimes \beta_1 \text{ or } \alpha_1 \otimes \beta_0]$ is at most the probability to project the state to either $S_1^\perp \times T_0^\perp$ or $S_0^\perp \times T_1^\perp$.

Let p_0 be the maximum among the probabilities to project the content of our two registers to $S_i^\perp \times T_j^\perp$, where $i, j \in \{0, 1\}$, and w.l.g., let the maximum be obtained for $i = j = 0$. Observe that in this case p_0 is just the probability to project the first register to S_0^\perp times the probability to project the second register to T_0^\perp . The former is equal to the probability to declare with certainty that “the first register contains α_1 ” and the latter is equal to the probability to declare “the second register contains β_1 ”, with certainty too. By definition, this is bounded above by ab , and therefore $p \leq 4p_0 \leq 4ab$. ■ *Lemma 15*